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Semi-Classical Calculation of the Particle Production in QED via the Schwinger Dynamical Principle

Jaume Haro[1]

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The Schwinger quantum dynamical principle is used to calculate the free vacuum persistence amplitude in the presence of a prescribed electromagnetic background, and the probability that free pairs are created from the vacuum state. An explicit expression of these amplitudes is obtained in the semi-classical approach, showing that, in this approach, the particle production is a stochastic Poisson process.

KEY WORDS: quantum electrodynamics; semi-classical limit; Schwinger dynamical principle.

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1. INTRODUCTION

External time-dependent electromagnetic backgrounds modify the structure of the quantum vacuum, what manifest in the production and annihilation of particles and antiparticles. Some authors have been calculated the vacuum persistence amplitude and the particle production in the case of an homogeneous electric field using several methods (see Nikishov, 1970; Popov, 1972; Marinov and Popov, 1977; Haro, 2003; Haro, 2004b, etc.). Here we are interested in the production of electrons and positrons in the presence of a no-homogeneous electric field (The bossonic case has been studied in Haro (2004a)).

To accomplish with this program, firstly we are interested to deduce, from the quantum electrodynamics in the Heisenberg picture, the so called Schwinger's dynamical principle. This principle shows the relation that exists between the first variation of the "Free Vacuum-to-Free Vacuum" amplitude and the Green's functions that appear in quantum electrodynamics. As an application of this dynamical

[1] Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain; e-mail: jaime.haro@upc.edu

principle, we obtain the formula that provides the free vacuum persistence amplitude.

Once we have obtained this probability, we deduce, using the Green's functions, the master formula that gives the probability that a prescribed number of free pairs is created from the free vacuum state.

Finally, to obtain an explicit expression of the master formula we use the Perturbation theory and the semi-classical approach. We show that the average number of produced pairs at time τ from the vacuum state by a prescribed electric field is

$$\frac{3\alpha}{32mc^2}\mathcal{E}(\tau),$$

where $\alpha \equiv \frac{e^2}{\hbar c}$ is the fine structure constant, and $\mathcal{E}(\tau)$ is the energy of the field at time τ.

And since, in the semi-classical limit, the creation of pairs is a stochastic Poisson process, we will conclude that the probability that N pairs are created at time τ is

$$\frac{1}{N!}\left(\frac{3\alpha}{32mc^2}\mathcal{E}(\tau)\right)^N \exp\left(-\frac{3\alpha}{32mc^2}\mathcal{E}(\tau)\right).$$

The paper is organized as follows. Section 2 is devoted to the definition of the vacuum state in the presence of an external electromagnetic field, and the definition of the so called "in" and "out" vacuum states. In Section 3, the Schwinger's dynamical principle is deduced using the Heisenberg and Interaction pictures, and the Green's functions are used to describe the pair production phenomenon. Finally, in Section 4 we calculate, using the semi-classical approach, the probability that N pairs are created from the vacuum state, and the average number of produced pairs.

In the paper we will use the following notation:

$$x \equiv x^\mu = (ct, \mathbf{x}); \quad p \equiv p^\mu = \left(\frac{E}{c}, \mathbf{p}\right); \quad A \equiv A^\mu = (V, \mathbf{A});$$

$$\partial \equiv \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right).$$

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1).$$

$|\Psi_S(t)\rangle \equiv |\Psi(t)\rangle$ denotes the state $|\Psi(t)\rangle$ in the Schrödinger picture.
$|\Psi_H(t)\rangle \equiv U_S^\dagger(t, 0)|\Psi(t)\rangle$ denotes the state $|\Psi(t)\rangle$ in the Heisenberg picture.
$|\Psi_I(t)\rangle \equiv U_0^\dagger(t, 0)|\Psi(t)\rangle$ denotes the state $|\Psi(t)\rangle$ in the Interaction picture,

where U_S is the quantum evolution operator and U_0 is the free quantum evolution operator.

In the same way:

$\hat{B}_S(t) \equiv \hat{B}(t)$ denotes the operator $\hat{B}(t)$ in the Schrödinger picture.
$\hat{B}_H(t) \equiv U_S^\dagger(t,0)\hat{B}(t)U_S(t,0)$ denotes the operator $\hat{B}(t)$ in the Heisenberg picture.
$\hat{B}_I(t) \equiv U_0^\dagger(t,0)\hat{B}(t)U_0(t,0)$ denotes the operator $\hat{B}(t)$ in the Interaction picture.

2. THE VACUUM STATE IN QUANTUM ELECTRODYNAMICS

Let $\hat{H}(t; A_\mu)$ be the quantum Hamiltonian operator. We assume that we have the decomposition $\hat{H}(t; A_\mu) = \hat{H}_K(t; \mathbf{A}) + \hat{H}_P(t; V)$, where $\hat{H}_K(t; \mathbf{A})$ is the kinetic energy operator and $\hat{H}_P(t; V)$ is the potential energy operator.

Example 2.1. For the Dirac field, we have (see Schwinger, 1951a; Dittrich and Reuter, 1984)

$$\hat{H}_K(t; \mathbf{A}) = \int_{\mathbb{R}^3} d^3\mathbf{x} \left(\frac{1}{2}[\hat{\bar{\psi}}(\mathbf{x}), (-i\hbar c\gamma\nabla + mc^2)\hat{\psi}(\mathbf{x})] - \frac{1}{c}\mathbf{A}(t,\mathbf{x})\hat{J}(\mathbf{x}) \right).$$

$$\hat{H}_P(t; V) = \frac{1}{c}\int_{\mathbb{R}^3} d^3\mathbf{x}\hat{J}^0(\mathbf{x})V(t,\mathbf{x}),$$

where $\hat{J}^\mu(\mathbf{x}) \equiv (\hat{J}^0(\mathbf{x}), \hat{J}(\mathbf{x})) = \frac{ec}{2}[\hat{\bar{\psi}}(\mathbf{x}), \gamma^\mu\hat{\psi}(\mathbf{x})] = -\frac{ec}{2}\gamma^\mu_{\beta\alpha}[\hat{\psi}_\alpha(\mathbf{x}), \hat{\bar{\psi}}_\beta(\mathbf{x})]$ is the symmetrized current operator in the Schrödinger picture.

Let $\lambda(t; \mathbf{A})$ be the minimum eigenvalue of $\hat{H}_K(t; \mathbf{A})$. Define now the renormalized quantum kinetic energy operator, $\hat{\hat{H}}_K(t; \mathbf{A}) \equiv \hat{H}_K(t; \mathbf{A}) - \lambda(t; \mathbf{A})Id$. Then, the vacuum state at time t, denoted by $|0^{\mathbf{A}}(t)\rangle$, satisfies (see Dirac, 1934; Grib *et al.*, 1994; Dolby and Gull, 2001)

$$\hat{\hat{H}}_K(t; \mathbf{A})|0^{\mathbf{A}}(t)\rangle = 0; \quad \hat{Q}|0^{\mathbf{A}}(t)\rangle = 0,$$

where $\hat{Q}$ is the charge operator.

Remark 2.1. In the case that $A_\mu = (V, \mathbf{0})$, the operator $\hat{\hat{H}}_K(t; \mathbf{0})$ do not depend on time, and it can be denoted by $\hat{\hat{H}}_K(\mathbf{0})$. Then the vacuum state, denoted by $|0^{\mathbf{0}}\rangle$, is independent on time and coincides with the free vacuum state.

Example 2.2. Consider the free Dirac field, and define

$$E_\pm(\mathbf{p}) = \pm\sqrt{c^2\mathbf{p}^2 + m^2c^4}; \quad p^\mu_\pm = \left(\frac{1}{c}E_\pm(\mathbf{p}), \mathbf{p}\right).$$

Taking the spinors with the following properties

$$(\gamma_\mu p^\mu_\pm - mc)u^\pm_\alpha(\mathbf{p}) = 0; \quad \alpha = 1, 2.$$

$$\bar{u}^\pm_\alpha(\mathbf{p})\gamma^\mu u^\pm_\beta(\mathbf{p}) = \pm 2cp^\mu_\pm \delta_{\alpha\beta}; \quad \bar{u}^\pm_\alpha(\mathbf{p})u^\pm_\beta(\mathbf{p}) = \pm 2mc^2\delta_{\alpha\beta},$$

we obtain the following decomposition of the Dirac field operator

$$\hat{\psi}(\mathbf{x}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\sum_{\sigma=1}^{2}\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{\sqrt{2E_+(\mathbf{p})}}(\hat{a}_\sigma(\mathbf{p})u^+_\sigma(\mathbf{p}) + \hat{b}^\dagger_{-\sigma}(-\mathbf{p})u^-_\sigma(\mathbf{p}))e^{\frac{i}{\hbar}\mathbf{p}.\mathbf{x}}.$$

Then, the free vacuum state satisfies

$$\hat{a}_\sigma(\mathbf{p})|0^{\mathbf{0}}\rangle = \hat{b}_\sigma(\mathbf{p})|0^{\mathbf{0}}\rangle = 0; \quad \forall\mathbf{p}\in\mathbb{R}^3, \quad \sigma = 1, 2,$$

because for the free Dirac field we have

$$\hat{\hat{H}}_K(\mathbf{0}) = \sum_{\sigma=1}^{2}\int_{\mathbb{R}^3} d^3\mathbf{p}E_+(\mathbf{p})(\hat{a}^\dagger_\sigma(\mathbf{p})\hat{a}_\sigma(\mathbf{p}) + \hat{b}^\dagger_\sigma(\mathbf{p})\hat{b}_\sigma(\mathbf{p})).$$

Consider now the following kinetic energy operator

$$\hat{H}_K(\mathbf{A}) = \frac{1}{2}\int_{\mathbb{R}^3} d^3\mathbf{x}[\hat{\bar{\psi}}(\mathbf{x})(\gamma\cdot(-i\hbar c\nabla - e\mathbf{A}) + mc^2)\hat{\psi}(\mathbf{x})],$$

where we have assumed that **A** is a constant vector independent on **x**.

Using the decomposition

$$\hat{\psi}(\mathbf{x}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\sum_{\sigma=1}^{2}\int_{\mathbb{R}^3}\frac{d^3\mathbf{p}}{\sqrt{2E_+(\mathbf{p}-\frac{e}{c}\mathbf{A})}}\Bigg(\hat{c}_\sigma(\mathbf{p})u^+_\sigma\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)$$

$$+\hat{d}^\dagger_{-\sigma}(-\mathbf{p})u^-_\sigma\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\Bigg)e^{\frac{i}{\hbar}\mathbf{p}.\mathbf{x}},$$

we can show that the renormalized kinetic energy operator is

$$\hat{\hat{H}}_K(\mathbf{A}) = \sum_{\sigma=1}^{2}\int_{\mathbb{R}^3} d^3\mathbf{p}E_+\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)(\hat{c}^\dagger_\sigma(\mathbf{p})\hat{c}_\sigma(\mathbf{p}) + \hat{d}^\dagger_\sigma(-\mathbf{p})\hat{d}_\sigma(-\mathbf{p})).$$

Then, the vacuum state, denoted by $|0^{\mathbf{A}}\rangle$, satisfies

$$\hat{c}_\sigma(\mathbf{p})|0^{\mathbf{A}}\rangle = \hat{d}_\sigma(\mathbf{p})|0^{\mathbf{A}}\rangle = 0; \quad \forall\mathbf{p}\in\mathbb{R}^3, \quad \sigma = 1, 2.$$

Note that we have the following relation

$$\hat{a}_\sigma(\mathbf{p}) = \frac{1}{2mc^2}\sqrt{\frac{E_+(\mathbf{p})}{E_+\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)}}\sum_{\beta=1}^{2}\Bigg(\hat{c}_\beta(\mathbf{p})\bar{u}^+_\sigma(\mathbf{p})u^+_\beta\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)$$

$$+\hat{d}^\dagger_{-\beta}(-\mathbf{p})\bar{u}^+_\sigma(\mathbf{p})u^-_\beta\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\Bigg)$$

$$\hat{b}^{\dagger}_{-\sigma}(-\mathbf{p}) = \frac{-1}{2mc^2}\sqrt{\frac{E_+(\mathbf{p})}{E_+\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)}}\sum_{\beta=1}^{2}\left(\hat{c}_\beta(\mathbf{p})\bar{u}^-_\sigma(\mathbf{p})u^+_\beta\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\right.$$
$$\left.+\hat{d}^{\dagger}_{-\beta}(-\mathbf{p})\bar{u}^-_\sigma(\mathbf{p})u^-_\beta\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\right).$$

Finally, it is easy to check that for $\mathbf{A} \neq \mathbf{0}$, we have

$$\bar{u}^+_\sigma(\mathbf{p})u^-_\beta\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right) \neq 0; \quad \bar{u}^-_\sigma(\mathbf{p})u^+_\beta\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right) \neq 0,$$

and consequently, $|0^{\mathbf{A}}\rangle \neq |0^{\mathbf{0}}\rangle$, that is, the vacuum state depends on the vector potential. This fact will be used in Section 3.1 in order to apply correctly the Schwinger dynamical principle to calculate the vacuum persistence amplitude.

To finish this section, we introduce the "in" and "out" vacuum states defined as follows:

$$\left|0^{A\mu}_{\text{in}}\right\rangle = \lim_{t\to-\infty}\left|0^{\mathbf{A}}_H(t)\right\rangle = \lim_{t\to-\infty} U^+_S(t,0)|0^{\mathbf{A}}(t)\rangle.$$
$$\left|0^{A\mu}_{\text{out}}\right\rangle = \lim_{t\to\infty}\left|0^{\mathbf{A}}_H(t)\right\rangle = \lim_{t\to\infty} U^+_S(t,0)|0^{\mathbf{A}}(t)\rangle.$$

3. THE QUANTUM DYNAMICAL PRINCIPLE

Let $|\Phi\rangle$ and $|\Psi\rangle$ be two quantum states and define $\Omega_{\tau',\tau} \equiv (c\tau', c\tau) \times \mathbb{R}^3$. We assume that the renormalized quantum Hamiltonian in the Schrödinger picture has the form

$$\hat{\hat{H}}_K(\mathbf{0}) + \frac{1}{c}\int_{\mathbb{R}^3} A_\mu(x)\hat{J}^\mu(\mathbf{x}), \tag{1}$$

where $\hat{\hat{H}}_K(\mathbf{0})$ denotes the renormalized free kinetic energy operator. Therefore the quantum evolution operator in the Interaction picture is

$$U_I(\tau,\tau') = \mathcal{T}\exp\left(-\frac{i}{\hbar c}\int_{\Omega_{\tau',\tau}} d^4y\frac{1}{c}A_\mu(y)\hat{J}^\mu_I(y)\right),$$

where we have introduced the time ordering operator $\mathcal{T}$.

If we introduce the potential $\tilde{A}_\mu = A_\mu\chi_{\Omega_{\tau',\tau}}$, where $\chi_{\Omega_{\tau',\tau}}$ is the characteristic function of the set $\Omega_{\tau',\tau}$, we can write

$$U_I(\tau,\tau') = \mathcal{T}\ \exp\left(-\frac{i}{\hbar c}\int_{\mathbb{R}^4} d^4y\frac{1}{c}\tilde{A}_\mu(y)\hat{J}^\mu_I(y)\right). \tag{2}$$

Consider now the functional $\mathcal{F}$, defined by

$$\mathcal{F}[\tilde{A}_\mu] \equiv \langle\Phi_H(\tau)|\Psi_H(\tau')\rangle = \langle\Phi_I(\tau)|U_I(\tau,\tau')|\Psi_I(\tau')\rangle. \tag{3}$$

The first variation of the functional $\mathcal{F}$ at the point $\tilde{A}_\mu$ in the direction of ϕ_μ is (see for detail Giaquinta and Hildebrandt, 1996)

$$\delta\mathcal{F}[\tilde{A}_\mu;\phi_\mu] = -\frac{i}{\hbar c}\int_{\mathbb{R}^4} d^4y\frac{1}{c}\phi_\mu(y)\big\langle\Phi_H(\tau)\big|\hat{J}^\mu_H(y)\big|\Psi_H(\tau')\big\rangle.$$

Now, introducing the functional $\mathcal{L}$ defined by

$$\mathcal{L}[\tilde{A}_\mu] \equiv -\frac{1}{c}\int_{\mathbb{R}^4} d^4y\frac{1}{c}\tilde{A}_\mu(y)\hat{J}^\mu_H(y),$$

we can find

$$\delta\mathcal{F}[\tilde{A}_\mu;\phi_\mu] = \frac{i}{\hbar}\langle\Phi_H(\tau)|\delta\mathcal{L}[\tilde{A}_\mu;\phi_\mu]|\Psi_H(\tau')\rangle. \tag{4}$$

This is the so called Schwinger's dynamical principle (see Schwinger, 1951b, formula 2.14; DeWitt, 1965). Now if we take the functional $\mathcal{W}[\tilde{A}_\mu] \equiv -i\hbar\,\log\,\mathcal{F}[\tilde{A}_\mu]$, we have

$$\begin{aligned}\delta\mathcal{W}[\tilde{A}_\mu;\phi_\mu] &= \frac{1}{\mathcal{F}[\tilde{A}_\mu]}\langle\Phi_H(\tau)|\delta\mathcal{L}[\tilde{A}_\mu;\phi_\mu]|\Psi_H(\tau')\rangle.\\ &= -\frac{1}{\mathcal{F}[\tilde{A}_\mu]}\frac{1}{c}\int_{\mathbb{R}^4} d^4y\frac{1}{c}\phi_\mu(y)\big\langle\Phi_H(\tau)\big|\hat{J}^\mu_H(y)\big|\Psi_H(\tau')\big\rangle.\end{aligned}$$

Finally, introducing the average current density $\langle\hat{J}^\mu_H(y)\rangle \equiv \frac{1}{F[\tilde{A}_\mu]}\langle\Phi_H(\tau)|$ $\hat{J}^\mu_H(y)|\Psi_H(\tau')\rangle$, we obtain the following relation

$$\delta\mathcal{W}[\tilde{A}_\mu;\phi_\mu] = -\frac{1}{c}\int_{\mathbb{R}^4} d^4y\frac{1}{c}\phi_\mu(y)\big\langle\hat{J}^\mu_H(y)\big\rangle. \tag{5}$$

This is the fundamental formula used by Schwinger to obtain an operational expression to the free vacuum persistence amplitude (see Schwinger, 1951a).

3.1. Free Vacuum-to-Free Vacuum transitions

Here we consider the Dirac field. Taking the symmetrized current density operator $\hat{J}^\mu_H = \frac{ec}{2}[\hat{\bar{\psi}}_H, \gamma^\mu\hat{\psi}_H]$, it is easy to check that

$$\big\langle\hat{J}^\mu_H(y)\big\rangle = -ie\,\hbar c\mathrm{Sp}(\gamma^\mu S[\tilde{A}_\mu](y,y)),$$

where $S[\tilde{A}_\mu](x,x') = \frac{-i}{\hbar F[\tilde{A}_\mu]}\langle\Phi_H(\tau)|\mathcal{T}\hat{\psi}_H(x)\hat{\bar{\psi}}_H(x')|\Psi_H(\tau')\rangle$ is a Green function of the Dirac field, $S[\tilde{A}_\mu](y,y) \equiv \frac{1}{2}\lim_{\mathbf{y}'\to\mathbf{y}}(\lim_{y_0'\to y_0^+} S[\tilde{A}_\mu](y,y') + \lim_{y_0'\to y_0}$ $-S[\tilde{A}_\mu](y,y'))$, and "Sp" means the trace in the spinor space.

Now, the problem is to determine this Green function. We are interested in the particular case that the initial and final states are the free vacuum state, that is, when $\mathcal{F}[\tilde{A}_\mu] = \langle 0^{\mathbf{0}}_H(\tau)|0^{\mathbf{0}}_H(\tau')\rangle$. In this situation, the Green function is

$$S[\tilde{A}_\mu](x,x') = -\frac{i}{\hbar}\frac{1}{\langle 0^{\mathbf{0}}_H(\tau)|0^{\mathbf{0}}_H(\tau')\rangle}\left\langle 0^{\mathbf{0}}_H(\tau)|\mathcal{T}\hat{\psi}_H(x)\hat{\bar{\psi}}_H(x')|0^{\mathbf{0}}_H(\tau')\right\rangle.$$

To determine this Green function, we write $S[\tilde{A}_\mu](x,x')$ in the following form

$$S[\tilde{A}_\mu](x,x') = -\frac{i}{\hbar}\frac{1}{\langle 0^{\mathbf{0}}|U_I(\tau,\tau')|0^{\mathbf{0}}\rangle}\langle 0^{\mathbf{0}}|\mathcal{T}\hat{\psi}_I(x)\hat{\bar{\psi}}_I(x')U_I(\tau,\tau')|0^{\mathbf{0}}\rangle,$$

where $U_I(\tau,\tau')$ is defined in formula (2). Then, expanding the exponential and using the Wick's Theorem we reach the following expression

$$S[\tilde{A}_\mu](x,x') = \langle x|\hat{S}[0]\sum_{n=0}^{\infty}\left(\frac{e}{c}\tilde{A}_\mu\gamma^\mu\hat{S}[0]\right)^n|x'\rangle,$$

where $\hat{S}[0] = (i\hbar\gamma^\mu\partial_\mu - mc + i\epsilon)^{-1}$ is the free Feynman propagator in the operator form. Finally, using the formula $(\hat{U}-\hat{V})^{-1} = \hat{U}^{-1}\sum_{n=0}^{\infty}(\hat{V}\hat{U}^{-1})^n$ (see Feynman, 1951), we obtain

$$S[\tilde{A}_\mu](x,x') = \langle x|\hat{S}[\tilde{A}_\mu]|x'\rangle = -i\int_0^\infty ds\langle x|\exp(is\,\hat{S}^{-1}[\tilde{A}_\mu])|x'\rangle, \qquad (6)$$

where $\hat{S}[\tilde{A}_\mu] \equiv (\gamma^\mu(i\hbar\partial_\mu - \frac{e}{c}\tilde{A}_\mu) - mc + i\epsilon)^{-1}$

Now we define,

$$\mathcal{V}_1[\tilde{A}_\mu] \equiv i\hbar\mathrm{Tr}\int_0^\infty\frac{ds}{s}\exp(is\,\hat{S}^{-1}[\tilde{A}_\mu]);\quad\text{and}\quad \mathcal{V}_2[\tilde{A}_\mu] \equiv i\hbar\mathrm{Tr}\,\log(\hat{S}[\tilde{A}_\mu]\hat{S}^{-1}[0]),$$

where we have introduced the trace in both spinor and configuration space. Then we have $\delta\mathcal{W}[\tilde{A}_\mu;\phi_\mu] = \delta\mathcal{V}_1[\tilde{A}_\mu;\phi_\mu] = \delta\mathcal{V}_2[\tilde{A}_\mu;\phi_\mu]$, and consequently we obtain

$$\mathcal{W}[\tilde{A}_\mu] = \mathcal{V}_1[\tilde{A}_\mu] - \mathcal{V}_1[0] = \mathcal{V}_2[\tilde{A}_\mu],$$

because $\mathcal{W}[0] = 0$.

Remark 3.1. When the initial and final states are the free vacuum state, the real part of the functional $\mathcal{W}[\tilde{A}_\mu]$ is called the "one-loop effective action" (see for details Dittrich and Reuter, 1984).

Finally, we can conclude that the formula that provides the free vacuum persistence amplitude is

$$\begin{aligned}\left\langle 0^{\mathbf{0}}_H(\tau)|0^{\mathbf{0}}_H(\tau')\right\rangle &= \exp\left(-\mathrm{Tr}\int_0^\infty\frac{ds}{s}(e^{is\,\hat{S}^{-1}[\tilde{A}_\mu]} - e^{is\,\hat{S}^{-1}[0]})\right)\\ &= \exp(-\mathrm{Tr}\,\log(\hat{S}[\tilde{A}_\mu]\hat{S}^{-1}[0])). \qquad (7)\end{aligned}$$

This is the famous result obtained by Schwinger (1951a) and explained in more detail in the Introduction of Schwinger (1953).

Note that, in the case that the potential vector $\mathbf{A}(x)$ vanishes when $x^0 \to \pm\infty$, from (7) and the definition of the "in" and "out" vacuum states, we get

$$\langle 0_{\text{out}}^{A^\mu} | 0_{\text{in}}^{A^\mu} \rangle = \exp(-\text{Tr}\,\log(\hat{S}[A_\mu]\hat{S}^{-1}[0])). \tag{8}$$

Remark 3.2. If we assume that $\mathbf{A}(x)$ vanishes when $x^0 \to -\infty$, and $\lim_{x^0 \to \infty} \mathbf{A}(x) = \bar{\mathbf{A}}$, where $\bar{\mathbf{A}}$ is a constant vector different from $\mathbf{0}$. Then, in this situation

$$\langle 0_{\text{out}}^{A^\mu} | 0_{\text{in}}^{A^\mu} \rangle \neq \exp(-\text{Tr}\,\log(\hat{S}[A_\mu]\hat{S}^{-1}[0])), \tag{9}$$

because $|0^{\mathbf{0}}\rangle \neq |0^{\bar{\mathbf{A}}}\rangle$ (see Example 2.2). That is, the vacuum persistence amplitude do not coincides with the free one.

3.2. Creation of Free Pairs

Here we study the free pair production from the free vacuum state. We want to know the relation that exists between the probability that a prescribed number of free pairs are created and the Green's functions.

Since we study free particles, we use the Interaction picture. Then, using the notation introduced in the Example 2.2, we have

$$\hat{\psi}_I(x) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \sum_{\sigma=1}^{2} \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{\sqrt{2E_+(\mathbf{p})}} (\hat{a}_\sigma(\mathbf{p}) u_\sigma^+(\mathbf{p}) e^{-\frac{i}{\hbar} p_+^\mu x_\mu} + \hat{b}_\sigma^\dagger(\mathbf{p}) u_{-\sigma}^-(-\mathbf{p}) e^{\frac{i}{\hbar} p_+^\mu x_\mu}).$$

Denoted by ::, the **normal product operator** for free operators, it is easy to verify that the current density can be written in the following form $\hat{J}_I^\mu = ec : \hat{\bar{\psi}}_I \gamma^\mu \hat{\psi}_I :$, and then, applying the Wick's theorem to the formula (2), we can obtain the master formula that relates the quantum evolution operator in the Interaction picture with the Green function $S[\tilde{A}_\mu](x_j, x_j')$ defined in formula (6) and with the "free vacuum-to-free vacuum" amplitude $e^{\frac{i}{\hbar}\mathcal{W}[\tilde{A}_\mu]}$ obtained in formula (7)

$$\begin{aligned} U_I(\tau, \tau') =: \exp\Bigg(\frac{i}{\hbar}\Bigg(\mathcal{W}[\tilde{A}_\mu] - \int_{\mathbb{R}^8} d^4x d^4x' \hat{\bar{\psi}}_I(x) \Bigg[\frac{e}{c}\tilde{A}_\mu(x)\gamma^\mu \delta(x - x') \\ + \frac{e}{c}\tilde{A}_\mu(x)\gamma^\mu S[\tilde{A}_\mu](x, x') \frac{e}{c}\tilde{A}_\nu(x')\gamma^\nu\Bigg] \hat{\psi}_I(x')\Bigg)\Bigg) : \end{aligned}$$

If we use the operator $\hat{I}[\tilde{A}_\mu]$ introduced in Schwinger (1953) formula 71 and defined by the identity $\hat{S}[\tilde{A}_\mu] \equiv \hat{S}[0] + \hat{S}[0]\hat{I}[\tilde{A}_\mu]\hat{S}[0]$, the master formula becomes

$$U_I(\tau, \tau') =: \exp\left(\frac{i}{\hbar}\left(\mathcal{W}[\tilde{A}_\mu] - \int_{\mathbb{R}^8} d^4x d^4x' \hat{\bar{\psi}}_I(x) \langle x | \hat{I}[\tilde{A}_\mu] | x' \rangle \hat{\psi}_I(x')\right)\right) : \tag{10}$$

Let E^N be the space generated by the states that contain N free pairs, that is, the space generated by the vectors

$$\left\{\prod_{j=1}^{N} \hat{a}^{\dagger}_{\alpha_j}(\mathbf{p}_j)\hat{b}^{\dagger}_{\beta_j}(\mathbf{q}_j)|0^{\mathbf{0}}\rangle; \quad \mathbf{p}_j, \mathbf{q}_j \in \mathbb{R}^3 \text{ and } \alpha_j, \beta_j \in \{1,2\}\right\}.$$

Let now $|\xi^N\rangle \in E^N$. We want to obtain an operational formula that relate the amplitude $\langle \xi^1_H(\tau)|0^{\mathbf{0}}_H(\tau')\rangle$ with the operator $\hat{I}[\tilde{A}_\mu]$. It is clear that $\langle \xi^N_H(\tau)|0^{\mathbf{0}}_H(\tau')\rangle = \langle \xi^N_I(\tau)|U_I(\tau,\tau')|0^{\mathbf{0}}\rangle$ because $|0^{\mathbf{0}}_I(\tau')\rangle = |0^{\mathbf{0}}\rangle$.

Therefore, using the formula (10) it is easy to check that

$$\left\langle \xi^N_H(\tau)\middle|0^{\mathbf{0}}_H(\tau')\right\rangle = \exp\left(\frac{i}{\hbar}\mathcal{W}[\tilde{A}_\mu]\right)\left\langle \xi^N_H(\tau)\middle|0^{\mathbf{0}}_H(\tau')\right\rangle_R,$$

where the relative amplitude $\langle \xi^N_H(\tau)|0^{\mathbf{0}}_H(\tau')\rangle_R$ is

$$\left\langle \xi^N_H(\tau)\middle|0^{\mathbf{0}}_H(\tau')\right\rangle_R = \frac{1}{N!}\left(-\frac{i}{\hbar}\right)^N \left\langle \xi^N_I(\tau)\right|$$
$$\times \int_{\mathbb{R}^{8N}} d^4x_1\cdots d^4x'_N : \prod_{j=1}^{N} \hat{\bar{\psi}}_I(x_j)\langle x_j|\hat{I}[\tilde{A}_\mu]|x'_j\rangle \hat{\psi}_I(x'_j) : |0^{\mathbf{0}}\rangle.$$

Remark 3.3. Note that in this formula, the normal ordering operator has the same effect that the time ordering operator.

We define now $|1^+_{\alpha_1}(\mathbf{p}_1)\cdots 1^-_{\beta_N}(\mathbf{q}_N)\rangle \equiv \prod_{j=1}^N \hat{a}^{\dagger}_{\alpha_j}(\mathbf{p}_j)\hat{b}^{\dagger}_{\beta_j}(\mathbf{q}_j)|0^{\mathbf{0}}\rangle$. Then an elementary calculation shows that

$$U_I(\tau,\tau')|0^{\mathbf{0}}\rangle = \exp\left(\frac{i}{\hbar}\mathcal{W}[\tilde{A}_\mu]\right)\Bigg[|0^{\mathbf{0}}\rangle + \sum_{N=1}^{\infty}\frac{1}{N!}\sum_{\alpha_1,\ldots,\beta_N=1}^{2}\int_{\mathbb{R}^{6N}}\prod_{J=1}^{N} b_{\alpha_j\beta_j}(\mathbf{p}_j,\mathbf{q}_j) \times |1^+_{\alpha_1}(\mathbf{p}_1)\cdots 1^-_{\beta_N}(\mathbf{q}_N)\rangle d^3\mathbf{p}_1\cdots d^3\mathbf{q}_N\Bigg], \tag{11}$$

with

$$b_{\alpha\beta}(\mathbf{p},\mathbf{q}) = -\frac{i}{\hbar}\int_{\mathbb{R}^8} d^4x d^4x' \bar{\psi}^+_{\alpha,\mathbf{p}}(x)\langle x|\hat{I}[\tilde{A}_\mu]|x'\rangle \psi^-_{\beta,\mathbf{q}}(x')$$

where we have introduced the notation

$$\bar{\psi}^+_{\alpha,\mathbf{p}}(x) \equiv \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\frac{1}{\sqrt{2E_+(\mathbf{p})}}\bar{u}^+_\alpha(\mathbf{p})e^{\frac{i}{\hbar}p^\mu_+ x_\mu}$$
$$\psi^-_{\beta,\mathbf{q}}(x') \equiv \frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\frac{1}{\sqrt{2E_+(\mathbf{q})}}u^-_{-\beta}(-\mathbf{q})e^{\frac{i}{\hbar}q^\mu_+ x'_\mu}.$$

Finally, with the aid of formula (11), it is easy to obtain an operational formula that gives the probability that N free pairs with prescribed momenta and spin are created form the free vacuum state.

Remark 3.4. For the Klein–Gordon field the formula corresponding to (11) is obtained in Haro (2004a).

4. PERTURBATIVE AND SEMI-CLASSICAL RESULTS

In this section we will calculate explicitly, in the first perturbative approximation, the probability that pairs are created in the presence of a prescribed no-homogeneous electric field. Assume that the potential $A^{\mu} = (V, \mathbf{0})$ vanishes when $x^0 \to -\infty$. In this situation the vacuum state coincides with the free one, and then, the probability that the vacuum state remains unchanged at time τ is $|\langle 0^0_H(\tau)|0^0_{\text{in}}\rangle|^2$. In order to calculate this probability, we use the following perturbative approximation (see formula (7))

$$-\text{Tr}\ \log(\hat{S}[\tilde{A}_{\mu}]\hat{S}^{-1}[0]) = \text{Tr}\log\left(Id. - \frac{e}{c}\gamma^{\mu}\tilde{A}_{\mu}\hat{S}[0]\right) \approx -\frac{e^2}{2c^2}\text{Tr}(\gamma^{\mu}\tilde{A}_{\mu}\hat{S}[0])^2.$$

Consequently, we have

$$\begin{aligned} \left|\langle 0^0_H(\tau)|0^0_{\text{in}}\rangle\right|^2 &= \exp(-2\text{Re Tr}\ \log(\hat{S}[\tilde{A}_{\mu}]\hat{S}^{-1}[0])) \\ &\approx \exp\left(-\frac{e^2}{c^2}\text{Re Tr}(\gamma^{\mu}\tilde{A}_{\mu}\hat{S}[0])^2\right), \end{aligned} \tag{12}$$

where $\tilde{A}_{\mu} = \chi_{(-\infty,\tau)\times\mathbb{R}^3}(V, \mathbf{0}) \equiv (\tilde{V}, \mathbf{0})$.

A elementary but cumbersome calculation provides the following result (see for details Schwinger, 1951a)

$$\begin{aligned} -\frac{e^2}{c^2}\text{Re Tr}(\gamma^{\mu}\tilde{A}_{\mu}\hat{S}[0])^2 = -\frac{e^2\pi}{6c^2\hbar^2}\int_{\mathbb{R}^4}\theta(\pi^2\hbar^2v^2 - m^2c^2)\mathbf{v}^2\mathfrak{T}_4\tilde{V}(v)\mathfrak{T}_4\tilde{V}(-v) \\ \times\sqrt{1 - \frac{m^2c^2}{\pi^2\hbar^2v^2}}\left(2 + \frac{m^2c^2}{\pi^2\hbar^2v^2}\right)d^4v, \end{aligned} \tag{13}$$

where, θ is the Heaviside step function, and $\mathfrak{T}_4 f(p) \equiv \int_{\mathbb{R}^4} f(x)e^{2\pi ipx}d^4x$ denotes the Fourier Transform of the function f.

Now, we calculate the formula (13) using the semi-classical approximation. This approximation is based in the following steps, first we make the change of variables $\pi\hbar v^0 = p$, thus

$$-\frac{e^2}{c^2}\mathrm{Re\,Tr}(\gamma^\mu \tilde{A}_\mu \hat{S}[0])^2$$
$$= -\frac{e^2}{6c^2\hbar^3}\int_{\mathbb{R}^4}\theta(p^2-\pi^2\hbar^2\mathbf{v}^2-m^2c^2)\mathbf{v}^2\mathfrak{T}_4\tilde{V}\left(\frac{p}{\pi\hbar},\mathbf{v}\right)\mathfrak{T}_4\tilde{V}\left(-\frac{p}{\pi\hbar},-\mathbf{v}\right)$$
$$\times\sqrt{1-\frac{m^2c^2}{p^2-\pi^2\hbar^2\mathbf{v}^2}}\left(2+\frac{m^2c^2}{p^2-\pi^2\hbar^2\mathbf{v}^2}\right)d^3\mathbf{v}\,dp. \tag{14}$$

Integrating by parts it is easy to check that

$$\mathfrak{T}_4\tilde{V}\left(\frac{p}{\pi\hbar},\mathbf{v}\right)\approx\frac{\hbar}{2ip}e^{\frac{2icp\tau}{\hbar}}\int_{\mathbb{R}^3}V(\tau,\mathbf{x})e^{-2\pi i\mathbf{v}\mathbf{x}}d^3\mathbf{x}\equiv\frac{\hbar}{2ip}e^{\frac{2icp\tau}{\hbar}}\mathfrak{T}_3V(\tau,\mathbf{v}).$$

Then, using this approximation and the properties, of the Fourier Transform, we obtain

$$-\frac{e^2}{c^2}\mathrm{Re\,Tr}(\gamma^\mu \tilde{A}_\mu \hat{S}[0])^2$$
$$= -\frac{e^2}{96\pi^2c^2\hbar}\int_{\mathbb{R}^4}\theta(p^2-\pi^2\hbar^2\mathbf{v}^2-m^2c^2)\mathfrak{T}_3\nabla V(\tau,\mathbf{v})\mathfrak{T}_3\nabla V(\tau,-\mathbf{v})$$
$$\times\frac{1}{p^2}\sqrt{1-\frac{m^2c^2}{p^2-\pi^2\hbar^2\mathbf{v}^2}}\left(2+\frac{m^2c^2}{p^2-\pi^2\hbar^2\mathbf{v}^2}\right)d^3\mathbf{v}\,dp. \tag{15}$$

Finally, in the semi-classical approximation, we must replace $\pi^2\hbar^2\mathbf{v}^2$ by 0, and we obtain, after integration

$$-\frac{e^2}{c^2}\mathrm{Re\,Tr}(\gamma^\mu \tilde{A}_\mu \hat{S}[0])^2\approx-\frac{3\alpha}{32mc^2}\mathcal{E}(\tau), \tag{16}$$

where $\alpha\equiv\frac{e^2}{\hbar c}$ is the fine structure constant, and $\mathcal{E}(\tau)\equiv\frac{1}{8\pi}\int_{\mathbb{R}^3}|\nabla V(\tau,\mathbf{x})|^2d^3\mathbf{x}$ is the energy of the electric field at time τ.

Once we have obtained this result, we can deduce that, using the perturbation theory and the semi-classical approximation, the probability that the vacuum state remains unchanged at time τ is

$$\left|\langle 0^0_H(\tau)|0^0_{\mathrm{in}}\rangle\right|^2\approx\exp\left(-\frac{3\alpha}{32mc^2}\mathcal{E}(\tau)\right).$$

Remark 4.1. In Haro (2003, 2004a), we have proved that, in the semi-classical approximation, the particle production is a stochastic Poisson process, then we can deduce that the probability that N pairs are produced at time τ is

$$\frac{1}{N!}\left(\frac{3\alpha}{32mc^2}\mathcal{E}(\tau)\right)^N \exp\left(-\frac{3\alpha}{32mc^2}\mathcal{E}(\tau)\right),$$

and the average number of produced pairs is $\frac{3\alpha}{32mc^2}\mathcal{E}(\tau)$.

Remark 4.2. Assume that the potential $(V, \mathbf{0})$ is switched on and off. Suppose that $\frac{\partial^{N-1}V}{\partial t^{N-1}}$ is a continuous function and suppose that $\frac{\partial^N V}{\partial t^N}$ has a jump discontinuity in the hyper-surface $T \times \mathbb{R}^3$. Then using the formula (14) and integration by parts, it is easy to prove that, the average number of produced pairs after the field is switched off, in the semi-classical approximation, is $\alpha\mathcal{O}(\hbar^{2N})$. In particular, for $N = 0$, the average number of produced pairs after the field is switched off is

$$\frac{3\alpha}{32mc^2}\lim_{\epsilon\to 0}\frac{1}{8\pi}\int_{\mathbb{R}^3}|\mathbf{E}(T+\epsilon,\mathbf{x}) - \mathbf{E}(T-\epsilon,\mathbf{x})|^2 d^3\mathbf{x},$$

where $\mathbf{E} \equiv -\nabla V$ is the electric field.

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